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Meromorphic and Holomorphic Approximation in C^m-Norms

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INTRODUCTION

Let Ω be a domain of the complex plane $\mathbb{C}$, and let F be a relatively closed subset of Ω. By $C(F)$, $H(F)$, $M(\Omega)$, and $M_F(\Omega)$ we denote, respectively, the space of all continuous functions on F, the functions holomorphic in a neighbourhood of F, the functions meromorphic in Ω, and the functions meromorphic in Ω with poles off F.

A usual procedure to investigate uniform approximation on closed sets consists in reducing the problem at hand to the analogous one for compact sets, through a well-known localization theorem which derives from Roth's fusion lemma [11].

Approximation problems in Lip α and BMO norms on closed subsets of $\mathbb{C}$ have been studied in [4] and [2], where obtaining an appropriate "fusion lemma" proves to be an essential too for characterizing those closed subsets F of Ω with the property that every function locally in lip α or VMO and holomorphic in the interior F^0 of F can be approximated, respectively, in Lip α or BMO norms by meromorphic functions. This paper addresses the problem of C^m-approximation, with $m \in \mathbb{R}^+$, from different sets of complex functions defined on arbitrary closed set of $\mathbb{C}$. Our interest focuses on two natural problems: approximation by functions in $H(F)$ and $M_F(\Omega)$, and approximation from $H(\Omega)$ (in particular, by entire functions).

By exploiting some ideas found in [4], we extend to closed sets previous results of Verdera and O'Farrell about C^m-approximation on compact sets (Corollary 6). On the other hand, we prove an Arakeljan-type theorem for C^m-norms (Theorem 11). These two are the main results of this paper. It should be remarked, however, that the C^m-fusion lemma (Theorem 4),

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whose proof we present in great detail, is of independent interest and reveals itself to be of capital importance in the study of other approximation problems to be treated in forthcoming papers.

Recently, by using constructive methods of localization of singularities, Boivin and Verdera [3] have proved a result similar to that established in Corollary 6, with $M_F(\Omega)$ replaced by $H(F)$. It is important to note that their result is actually equivalent to Corollary 6. In fact, Corollary 7 states that any function on $H(F)$ may be approximated in C^m-norms by meromorphic functions having no poles on F.

This paper is structured as follows. Preliminary results are gathered in Section 1. In Section 2 we give a short proof of Runge's Theorem in C^m-norms, where the poles of the approximating rational functions are fixed without affecting the approximation itself. An adequate and complete development of the C^m-fusion lemma is given in Section 3. Section 4 is devoted to showing that the problem of C^m-approximation by meromorphic functions is local (Theorem 5). The last section, Section 5, deals with C^m-approximation by holomorphic functions, and the above-mentioned extension of Arakeljan's Theorem is obtained.

1. Preliminaries

Let Ω be an open subset of $\mathbb{C}$, F a relatively closed set in Ω, and f a bounded complex function on F. The modulus of continuity, w_f, of f is defined by

$$w_f(r) = \sup\{|f(x) - f(y)| : x, y \in F, |x-y| \leqslant r, r \geqslant 0\}.$$

For $0 < \alpha < 1$ define

$$\|f\|_{\alpha, F} = \sup\{r^{-\alpha} w_f(r) : r > 0\}$$

$$\mathrm{Lip}(\alpha, F) = \{f : \|f\|_{\alpha, F} < \infty\}$$

$$\mathrm{lip}(\alpha, F) = \{f \in \mathrm{Lip}(\alpha, F) : r^{-\alpha} w_f(r) \to 0, \quad \text{as} \quad r \to 0^+\}$$

$$\mathrm{lip}_{\mathrm{loc}}(\alpha, F) = \{f \in \mathrm{lip}(\alpha, F \cap K), \forall K \text{ compact}, K \subset F\}.$$

We use the well known differential operators $\bar{\partial}$ and ∂:

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Now, let $m \in \mathbb{R}$, $m \geqslant 1$, and $\alpha = m - [m]$, where $[m]$ is the integer part of m. If $\alpha \neq 0$, denote by $C^m(\Omega)$ the set of complex functions on Ω, with bounded continuous derivatives up to order $[m]$, and for $i+j=[m]$, $\partial^i\bar{\partial}^j f \in \operatorname{lip}(\alpha, \Omega)$, endowed with the norm

$$\|f\|_{m,\Omega} = \sum_{0\leq i+j\leq [m]} \|\partial^i\bar{\partial}^j f\|_{\infty,\Omega} + \sum_{i+j=[m]} \|\partial^i\bar{\partial}^j f\|_{\alpha,\Omega},$$

where $\|g\|_{\infty,\Omega} = \sup\{|g(z)|, z\in\Omega\}$. $\|f\|_{m,\Omega}$ is called the C^m norm of f.

On the other hand, if $\alpha=0$ $(m\in\mathbb{N})$, $C^m(\Omega)$ is the space of those functions on Ω with bounded continuous derivatives up to order m, and its norm is given by

$$\|f\|_{m,\Omega} = \sum_{0\leq i+j\leq m} \|\partial^i\bar{\partial}^j f\|_{\infty,\Omega}.$$

In order to consider approximation of functions defined on closed sets we denote by $\mathscr{C}^m(\Omega)$ the class of functions on Ω with continuous derivatives up to order $[m]$ and for $i+j=[m]$, $\partial^i\bar{\partial}^j f \in \operatorname{lip}_{\mathrm{loc}}(\alpha, \Omega)$.

For a closed set F and a set A of functions defined on F, $A\subset \mathscr{C}^m(\Omega)$, we introduce

$$[A]_{m,F} = \{f\in\mathscr{C}^m(\Omega) : \exists f_n\in A, \|f-f_n\|_{m,F}\to 0, n\to\infty\},$$

where

$$\|f\|_{m,F} = \sum_{0\leq i+j\leq [m]} \|\partial^i\bar{\partial}^j f\|_{\infty,F} + \sum_{i+j=[m]} \|\partial^i\bar{\partial}^j f\|_{\alpha,F}. \tag{*}$$

Usually, there are several equivalent ways of thinking of the space $C^m(F)$ ($\operatorname{Lip}(m, F)$). Here it is simplest to think of it as the space of all functions on F which have extension in $C^m(\mathbb{C})$ [12, Chap. VI]. The norm of an element $f\in C^m(F)$ is

$$\|f\|_{m,F} = \inf\{\|g\|_{m,\mathbb{C}} : g\in C^m(\mathbb{C}), g=f \text{ on } F\},$$

and if $A\subset C^m(\mathbb{C})$ then the closure in $C^m(F)$ of A coincides with $[A]_{m,F}$. Therefore, provided that f is differentiable on a neighbourhood of F we are able to estimate the norm $\|f\|_{m,F}$ given in (*).

For a infinitely differentiable function $\phi: \mathbb{C}\to\mathbb{C}$ with compact support, the Vitushkin localization operator T_ϕ is defined by

$$(T_\phi f)(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z)-f(\xi)}{\xi - z}\frac{\partial\phi}{\partial\bar{\xi}}\,dm_\xi = f(z)\,\phi(z) + \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z)}{\xi - z}\frac{\partial\phi}{\partial\bar{\xi}}\,dm_\xi$$

for any function $f: \mathbb{C}\to\mathbb{C}$ and any $z\in\mathbb{C}$ such that the right-hand side exists. Here m is the two-dimensional Lebesgue measure. If f is locally $L^1(m)$, then $T_\phi f$ exists everywhere, and is analytic wherever f is analytic and off the support of ϕ. Also $f - T_\phi f$ is analytic on the interior of the set $\phi^{-1}(1)$ ([6, p. 30]).

For any subset A of a domain Ω ($\Omega \subseteq \mathbb{C}$), the interior, the closure, and the boundary of A relative to Ω are represented by A^0, $\bar{A}$, and ∂A, respectively. Also, $\Omega^* = \Omega \cup \{*\}$ denotes the one-point compactification of Ω (if $\Omega = \mathbb{C}$, $\mathbb{C}^* \cup \{\infty\}$), and we adopt the usual convention of denoting by M a positive constant, independent of the relevant variables under consideration, which may be different at each occurrence.

When X is a compact subset of $\mathbb{C}$, we denote by $R(X)$ the set of rational functions with poles off X and by $\mathscr{R}(X)$ the class of those functions belonging to $C^m(\mathbb{C})$, $m \in \mathbb{R}^+$, that agree on some neighbourhood of X with a function of $R(X)$. We conclude this section by recalling the result in Theorem 1, which is due to Verdera for $m \in \mathbb{N}$, and to O'Farrell for any m.

THEOREM 1 ([9], [13]). *Let X be a compact subset of $\mathbb{C}$ and $f \in C^m(\mathbb{C})$, $m \in \mathbb{R}^+$. Then the following statements are equivalent.*

(a) *There exists a sequence of functions $f_n \in \mathscr{R}(X)$ such that $\| f - f_n \|_{m, X} \to 0$.*

(b) $\partial^i \bar{\partial}^{j+1} f(z) = 0$ $\forall z \in X$, $0 \leqslant i + j \leqslant m - 1$.

2. A Proof of Runge's Theorem for C^m Norms

If X is a compact subset of $\mathbb{C}^*$, any $f \in H(X)$ can be approximated uniformly on X by rational functions without poles on X; we can even fix the poles of the approximating rational functions (Classical Runge's Theorem, [5]). By using methods of functional analysis with the Cauchy transform one has similar results in Lip α norms ([7, 4]).

The Runge Theorem for C^m norms on compact set of $\mathbb{C}$ can be deduced trivially from Theorem 1. However, let us expose a different proof where we can fix the poles of the approximating functions and consider compact subsets of $\mathbb{C}^*$ (instead of $\mathbb{C}$). This is essential in order to prove the C^m-fusion lemma and the pole-shifting theorem.

Let X be a compact set of $\mathbb{C}$ and denote by $\mathscr{H}(X)$ those functions of $C^m(\mathbb{C})$ which coincide on a neighbourhood of X with some function of $H(X)$. If $T \in (C^m(\mathbb{C}))'$ (the topological dual of $C^m(\mathbb{C})$), Runge's Theorem implies that T annihilates $\mathscr{R}(X)$ if and only if T annihilates $\mathscr{H}(X)$. Also by the Withney Extension Theorem [12, Chap. VI] we can identify $\mathscr{R}(X)$ and $\mathscr{H}(X)$ with $R(X)$ and $H(X)$, respectively. Hence, we have $[R(X)]_{m,X} = [H(X)]_{m,X}$ by the separation theorem (see also [7, p. 376]). Moreover, if $R_A(X)$ is the subspace of those f in $R(X)$ with poles on $A = \{\alpha_j\}_{j \in J}$, where each component of $\mathbb{C}^* \backslash X$ contains an α_j, then $[R_A(X)]_{m,X} = [H(X)]_{m,X}$.

Suppose now that K is a compact subset of $\mathbb{C}^*$, and f is an holomorphic function in a neigboorhood of K. If $\infty \in K$, for $a \in \mathbb{C}^* \setminus K$ we use the transformation

$$r_a(w) = \frac{1}{w-a}, \qquad w \in K.$$

Let $K_1 = r_a(K)$ and g be the function defined in K_1 as $g(z) = f(w)$, where $r_a(w) = z$. Of course, g is holomorphic in a neighbourhood of K_1.

It is enough to deal with the case $1 \leqslant m < 2$. Then by Runge Theorem in C^m-norms for a compact set of $\mathbb{C}$, there exists $r \in R(K_1)$ such that

$$\|g - r\|_{\infty, k_1} < \varepsilon \tag{1}$$

$$\left\| \frac{\partial g}{\partial z} - \frac{\partial r}{\partial z} \right\|_{\infty, K_1} < \varepsilon \tag{2}$$

and, if $\alpha \neq 0$,

$$\left\| \frac{\partial g}{\partial z} - \frac{\partial r}{\partial z} \right\|_{\alpha, K_1} < \varepsilon. \tag{3}$$

By (1) we have

$$\left| f(w) - r\left(\frac{1}{w-a}\right) \right| < \varepsilon, \qquad \forall w \in K,$$

and $R(w) = r(1/(w-a))$ is a rational function without poles in K, because $r(z)$ is a rational function without poles in K_1. Since

$$\frac{\partial f}{\partial w} - \frac{\partial R}{\partial w} = -\frac{1}{(w-a)^2} \left(\frac{\partial g}{\partial z} - \frac{\partial r}{\partial z} \right),$$

by making use of (2), we have

$$\left\| \frac{\partial f}{\partial w} - \frac{\partial R}{\partial w} \right\|_{\infty, K} = \left\| z^2 \left(\frac{\partial g}{\partial z} - \frac{\partial r}{\partial z} \right) \right\|_{\infty, K} \leqslant M\varepsilon$$

Finally by (3), if $w_1, w_2 \in K$ then

$$\begin{aligned} &\frac{\left| \left(\frac{\partial f}{\partial w} - \frac{\partial R}{\partial w} \right)(w_1) - \left(\frac{\partial f}{\partial w} - \frac{\partial R}{\partial w} \right)(w_2) \right|}{|w_1 - w_2|^\alpha} \\ &= \frac{\left| -z_1^2 \left(\frac{\partial g}{\partial z} - \frac{\partial r}{\partial z} \right)(z_1) + z_2^2 \left(\frac{\partial g}{\partial z} - \frac{\partial r}{\partial z} \right)(z_2) \right| |z_1 z_2|^\alpha}{|z_1 - z_2|^\alpha} \\ &\leqslant \|z^{2\alpha}\|_{\infty, K_1} \|z^2\|_{\alpha, K_1} \left\| \frac{\partial g}{\partial z} - \frac{\partial r}{\partial z} \right\|_{\alpha, K_1} < M_1 \varepsilon. \end{aligned}$$

We also remark that if the set $A = \{\alpha_j\}$ is fixed, then we can choose the function r with poles in $A_1 = \{r_a(\alpha_j)\}$. Therefore R has its poles in A. Thus, we get the most general form of Runge's Theorem in C^m norms:

THEOREM 2 (C^m–Runge Theorem). *Let K be a compact subset of $\mathbb{C}^*$ and let $A \subset \mathbb{C}^* \backslash K$ such that A meets every component of $\mathbb{C}^* \backslash K$. Then*

$$[H(K)]_{m,K} = [R(K)]_{m,K}$$

and

$$[H(K)]_{m,K} = [R_A(K)]_{m,K}.$$

3. THE FUSION LEMMA IN C^m-NORMS

This section also deals with approximation on compact sets. However, the C^m-fusion lemma that we prove here serves as a stepping stone to the study of C^m-approximation on noncompact sets, which is taken up in the next sections.

In proving Theorem 4 we exploit our ideas in [4, Theorem 3] for Lip α norms. Though some steps are similar, we preferred to show them in detail for the sake of completeness. When $f \in C^m(E)$ we need the following estimate:

PROPOSITION 3. *Let E be a closed subset of $\mathbb{C}$ and $f \in C^m(E)$. If $\alpha = m - [m]$ and $\alpha \neq 0$, then $\|\partial^i \bar{\partial}^j f\|_{\alpha,E} \leq C \|f\|_{m,E}$, whenever $i + j < [m]$, C being a constant independent of f and E.*

Proof. It is enough to work with $m = 1 + \alpha$; the general case then follows easily. Let $f \in C^m(E)$, and assume that f has an extension $f^* \in C^m(\mathbb{C})$ such that $\|f^*\|_{m,\mathbb{C}} \leq C \|f\|_{m,E}$ ([12, p. 177]).

If $x, y \in E$, by the mean value theorem applied to f^* on $\mathbb{C}$, we have

$$|f(x) - f(y)| \leq |x - y| \sum_{0 \leq i+j \leq 1} \|\partial^i \bar{\partial}^j f^*\|_{\infty,\mathbb{C}} \leq \|f^*\|_{m,\mathbb{C}} |x - y|,$$

so that $f \in \mathrm{Lip}(1, E)$ and $\|f\|_{\mathrm{Lip}\,1,E} \leq \|f\|_{m,\mathbb{C}} \leq C \|f\|_{m,E}$ (we denote by

$$\|f\|_{\mathrm{Lip}\,1,E} = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|}, x, y \in E \right\} \Big).$$

To estimate $\|f\|_{\alpha,E}$ we note that

$$\frac{|f(x) - f(y)|}{|x - y|^{\alpha}} = \frac{|f(x) - f(y)|}{|x - y|} |x - y|^{1-\alpha} \leq \|f\|_{\mathrm{Lip}\,1,E} |x - y|^{1-\alpha},$$

and we have the following two cases:

(a) if $|x-y|<1$, $\|f\|_{\alpha, E} \leqslant \|f\|_{\mathrm{Lip}\,1, E} \leqslant C\|f\|_{m, E}$, and

(b) if $|x-y| \geqslant 1$, $\|f\|_{\alpha, E} \leqslant 2\,\|f\|_{\infty, E} \leqslant C\|f\|_{m, E}$.

We can now state.

THEOREM 4 (The C^m-Fusion Lemma). *Suppose K_1, K_2, K are compact sets in the extended complex plane $\mathbb{C}^*$ such that $K_1 \cap K_2 = \varnothing$. Then there exist a constant M, depending only on K_1 and K_2, and a rational function r, such that if r_1 and r_2 are rational functions with $\|r_1 - r_2\|_{m, K} < \varepsilon$, then*

$$\|r - r_i\|_{m, K \cup K_i} < M\varepsilon \qquad (i = 1, 2).$$

Proof. Note that Theorem 4 follows immediately from Runge's Theorem when $K \cap K_1 = \varnothing$ or $K \cap K_2 = \varnothing$. Suppose, for example, that $K \cap K_1 = \varnothing$; we write

$$f(z) = \begin{cases} r_1(z) & (z \in K_1) \\ r_2(z) & (z \in K_2 \cup K). \end{cases}$$

Let H_1 denote the sum of the principal parts of r_1 on K_1, and let H_2 denote the sum of the principal parts of r_2 on $K_2 \cup K$. Then $f - H_1 - H_2$ is analytic on $K_1 \cup K_2 \cup K$, and by the C^m-Runge Theorem there exists a rational function R such that

$$\|f - H_1 - H_2 - R\|_{m, K \cup K_1 \cup K_2} < \varepsilon.$$

Thus the assertion of the theorem holds with $r = H_1 + H_2 + R$ and $M = 2$. Therefore we can suppose that $K \cap K_1 \neq \varnothing$ and $K \cap K_2 \neq \varnothing$. Moreover we can make the following assumptions:

(i) We can assume that $\infty \in K_2$ and $0 < \|r_1 - r_2\|_{m, K} < \infty$,

(ii) it is enough to deal with the case $r_2 \equiv 0$. For in the general case we let $\rho_1 = r_1 - r_2$, $\rho_2 = 0$, and there exists a rational function ρ such that

$$\|\rho - \rho_1\|_{m, K \cup K_1} < M\varepsilon$$

and

$$\|\rho\|_{m, K \cup K_1} < M\varepsilon.$$

That is,

$$\|(\rho + r_2) - r_1\|_{m, K \cup K_1} < M\varepsilon$$

and

$$\|(\rho+r_2)-r_2\|_{m,\,K\cup K_2}<M\varepsilon.$$

Hence in the sequel we suppose that $K_2\cap K\neq\varnothing$, $r_2\equiv 0$, $r_1\not\equiv 0$, $0<\|r_1\|_{m,K}<\infty$, and $\{\infty\}\in K_2$. Choose neighbourhoods U_i and $\hat{U}_i$ of K_i for $i=1,2$ and U of K, such that

(a) ∂U_i and $\partial\hat{U}_i\in\mathscr{C}^1$, $i=1,2$.

(b) $\bar{U}_i\subset\hat{U}_i$, $i=1,2$.

(c) $\bar{\hat{U}}_1\cap\bar{\hat{U}}_2=\varnothing$.

(d) $\|r_1\|_{m,\bar{U}}<4\,\|r_1\|_{m,K}$.

Let $E=\mathbb{C}\backslash(U_1\cup U_2)$. We extend the function $r_{1|\bar{U}\cap E}$ to $\mathbb{C}$ as a function ϕ in $C^{m+1}(\mathbb{C})$ such that

$$\|\phi\|_{m,E}\leqslant C\,\|r_1\|_{m,K},$$

with C a universal constant.

We introduce the function f defined on $\mathbb{C}$ by

$$f(z)=\begin{cases}\phi(z) & (z\in E)\\ r_1(z) & (z\in\mathbb{C}\backslash E)\end{cases}$$

Let φ be a infinitely differentiable function with compact support such that $\varphi_{|\hat{U}_1}\equiv 1$, $\varphi_{|\hat{U}_2}\equiv 0$, and $0\leqslant\varphi\leqslant 1$, for all $z\in\mathbb{C}$, and we define

$$F(z)=(T_\varphi f)(z)=\frac{1}{\pi}\int_{\mathbb{C}}\frac{f(z)-f(\xi)}{\xi-z}\frac{\partial\varphi}{\partial\bar{\xi}}\,\mathrm{dm}_\xi=f(z)\,\varphi(z)+g(z),$$

where

$$g(z)=\frac{1}{\pi}\int_{\mathbb{C}}\frac{f(z)}{\xi-z}\frac{\partial\varphi}{\partial\bar{\xi}}\,\mathrm{dm}_\xi.$$

From the properties of the Vitushkin operator, F is holomorphic in $K_1\cup K_2\cup K$, except for finitely many poles in K_1. Hence the theorem is proved if we show that

$$\|F-r_1\|_{m,\,K\cup K_1}<M\,\|r_1\|_{m,K} \tag{4}$$

$$\|F-r_2\|_{m,\,K\cup K_2}=\|F\|_{m,\,K\cup K_2}<M\,\|r_1\|_{m,K}, \tag{5}$$

where M is a constant which depends only on K_1 and K_2, since by applying the Runge Theorem in C^m-norms to the functions F–Σ (where Σ is the sum of the principal parts of F in U_1) there exists a rational function r_3 such that

$$\|F-r_3\|_{m,\,K\cup K_1\cup K_2}< M\,\|r_1\|_{m,\,K}.$$

We prove (4) and (5) by distinguishing two cases.

First, let $m\in\mathbb{N}$. In this case we have

$$\|g\|_{m,\,\mathbb{C}}\leqslant M\,\|r_1\|_{m,\,K}.$$

Indeed,

$$\|g\|_{m,\,\mathbb{C}}=\sum_{0\leqslant i+j\leqslant m}\|\partial^i\bar\partial^j g\|_{\infty,\,\mathbb{C}}=\sum_{0\leqslant i+j\leqslant m}\left\|\partial^i\bar\partial^j\frac{1}{\pi}\left(f\frac{\partial\varphi}{\partial\bar z}*\frac{1}{z}\right)\right\|_{\infty,\,\mathbb{C}}=(\mathrm{I}).$$

Note that if $z\in\mathbb{C}$,

$$\left|\partial^i\bar\partial^j\frac{1}{\pi}\left(f\frac{\partial\varphi}{\partial\bar z}*\frac{1}{z}\right)(z)\right|=\left|\frac{1}{\pi}\int_E\frac{\partial^i\bar\partial^j(f\,\bar\partial\varphi)}{\xi-z}\,\mathrm{dm}_\xi\right|\leqslant M\,\|\partial^i\bar\partial^j(f\,\bar\partial\varphi)\|_{\infty,\,E},$$

hence

$$(\mathrm{I})\leqslant M\,\|\varphi\|_{m+1,\,\mathbb{C}}\,\|f\|_{m,\,E}\leqslant M\,\|r_1\|_{m,\,K}.$$

Now, for $z\in K_1$, we have $F-r_1=g$ so that

$$\|F-r_1\|_{m,\,K_1}=\|g\|_{m,\,K_1}\leqslant M\,\|r_1\|_{m,\,K},\tag{6}$$

and for $z\in K$

$$\|F-r_1\|_{m,\,K}\leqslant\|(\varphi-1)\,r_1\|_{m,\,K}+\|g\|_{m,\,\mathbb{C}}\leqslant M\,\|r_1\|_{m,\,K}.\tag{7}$$

Clearly (6) and (7) show (4). Analogously, we have $F-r_2=F=g$ on K_2, so that

$$\|F\|_{m,\,K_2}=\|g\|_{m,\,K_2}\leqslant M\,\|r_1\|_{m,\,K},$$

and for $z\in K$,

$$\|F\|_{m,\,K}=\|\varphi r_1\|_{m,\,K}+\|g\|_{m,\,K}\leqslant M\,\|r_1\|_{m,\,K},$$

and (5) is also verified.

Suppose now that m is not an integer, then we work with $m=1+\alpha$. The general case follows from this.

We can write

$$\|g\|_{m,\mathbb{C}} = \sum_{0 \leqslant i+j \leqslant 1} \|\partial^i \bar{\partial}^j g\|_{\infty,\mathbb{C}} + \sum_{i+j=1} \|\partial^i \bar{\partial}^j g\|_{\alpha,\mathbb{C}} \leqslant (\mathrm{I}) + (\mathrm{II})$$

As in the first case we get

$$(\mathrm{I}) = M \|r_1\|_{m,K},$$

and by taking into account the estimate of the Cauchy transform in Lip α norms [4], we have for all i, j such that $i+j=1$

$$\|\partial^i \bar{\partial}^j g\|_{\alpha,\mathbb{C}} \leqslant M \|f\|_{m,E},$$

since, for example if $i=1, j=0$,

$$\|\partial g\|_{\alpha,\mathbb{C}} = \left\| \frac{1}{\pi} \int_E \frac{\partial(f\, \partial\bar{\varphi})}{\xi - z} \, \mathrm{dm}_\xi \right\|_{\alpha,\mathbb{C}}$$

and, if $r = |x-y|/2$,

$$\begin{aligned} |\partial g(x) - \partial g(y)| &\leqslant \left| \frac{1}{\pi} \int_E \frac{x-y}{(\xi - x)(\xi - y)} \, \partial(f\, \bar{\partial}\varphi) \, \mathrm{dm}_\xi \right| \\ &\leqslant \frac{2r}{\pi} \left(\|\partial f\, \bar{\partial}\varphi\|_{\alpha,E} + \|f\, \partial\bar{\partial}\varphi\|_{\infty,E} \right) \\ &\quad \times \int_{\operatorname{supp} \bar{\partial}\varphi} \frac{\mathrm{dm}_\xi}{|\xi - x|\, |\xi - y|} = (\mathrm{III}), \end{aligned}$$

since both $\partial f\, \bar{\partial}\varphi$ and $f\, \partial\bar{\partial}\varphi$ vanish at one point in E ($\operatorname{supp} \bar{\partial}\varphi \subset \bar{U}_2 \setminus \hat{U}_1$); then

$$\|\partial f\, \bar{\partial}\varphi\|_{\infty,E} < M (\operatorname{diam} E)^\alpha \|\partial f\|_{\alpha,E} < M \|f\|_{m,E}$$

and from Proposition 3

$$\|f\, \partial\bar{\partial}\varphi\|_{\infty,E} < M (\operatorname{diam} E)^\alpha \|f\|_{\alpha,E} < M \|f\|_{m,E}.$$

By proceeding as in [4] (or [10])

$$(\mathrm{III}) \leqslant M \|f\|_{m,E} |x-y|^\alpha$$

and

$$(\mathrm{II}) \leqslant M \|r_1\|_{m,K}.$$

Hence one has

$$\|g\|_{m,\mathbb{C}} \leqslant M\,\|r_1\|_{m,K}.$$

In order to verify (4), we have

$$\|F-r_1\|_{m,K\cup K_1} \leqslant \|f\varphi - r_1\|_{m,K\cup K_1} + \|g\|_{m,\mathbb{C}}$$

where

$$\|f\varphi - r_1\|_{m,K\cup K_1} = \|f\varphi - r_1\|_{1,K\cup K_1} + \sum_{i+j=1} \|\partial^i\bar{\partial}^j(f\varphi - r_1)\|_{\alpha,K\cup K_1}$$

and we can estimate the first addend as in the integer case for $m=1$. For the second one, note that $f\varphi - r_1 = (\varphi - 1)\,r_1$ on $K\cup K_1$, and

$$\partial((\varphi-1)\,r_1) = \partial r_1(\varphi-1) + r_1\partial(\varphi-1) = \partial r_1(\varphi-1) + r_1\partial\varphi \tag{8}$$

$$\bar{\partial}((\varphi-1)\,r_1) = r_1\bar{\partial}(\varphi-1) = r_1\bar{\partial}\varphi. \tag{9}$$

Let $l = r_1\partial\varphi$. We must consider the following cases:

(i) If $x, y\in K$, then

$$\|r_1\partial\varphi\|_{\alpha,K} \leqslant \|\partial\varphi\|_{\alpha,\mathbb{C}}\,\|r_1\|_{\alpha,K} \leqslant M\,\|r_1\|_{m,K}.$$

(ii) If $x, y\in K_1$, $l\equiv 0$, therefore $\|l\|_{\alpha,K_1} = 0$.

(iii) Finally, if $x\in K$ and $y\in K_1$, since l vanishes on K_1

$$\frac{|l(x)-l(y)|}{|x-y|^\alpha} = \frac{|l(x)|}{|x-y|^\alpha}.$$

Also note that l vanishes at some point u of $K\cap E$. Moreover, we can assume that $x\in U_1^c\cap K$ (if $x\in U_1$ then $l(x)=0$) and there exists a $w\in U$ such that $|x-w|\leqslant|x-y|$. Hence

$$\begin{aligned}
\frac{|l(x)-l(y)|}{|x-y|^\alpha} &\leqslant \frac{|l(x)-l(w)|}{|x-w|^\alpha} + \frac{|l(w)|}{|x-y|^\alpha}\\
&\leqslant \|l\|_{\alpha,\bar{U}} + \|l\|_{\alpha,\bar{U}}\frac{|w-u|^\alpha}{|x-y|^\alpha}\\
&\leqslant \|l\|_{\alpha,\bar{U}}\left(1 + \frac{(|w-y|+|y-u|)^\alpha}{|x-y|^\alpha}\right)\\
&\leqslant M\left(1+\left(1+\frac{d}{\beta}\right)^\alpha\right)\|r_1\|_{\alpha,K} \leqslant M\,\|r_1\|_{m,K},
\end{aligned}$$

where the last estimation follows from Proposition 3, $\beta = \text{dist}(U_1^c, K_1)$ and $d = \text{diam } E$. Similar estimations follow for $l = \partial r_1(\varphi - 1)$ and $l = r_1 \bar{\partial}\varphi$, and (4) holds.

Analogously, in order to verify (5), we have

$$\|F\|_{m, K \cup K_2} \leqslant \|f\varphi\|_{m, K \cup K_2} + \|g\|_{m, \mathbb{C}},$$

where

$$\|f\varphi\|_{m, K \cup K_2} = \|f\varphi\|_{1, K \cup K_2} + \sum_{i+j=1} \|\partial^i \bar{\partial}^j (f\varphi)\|_{\alpha, K \cup K_2}$$

and we can estimate the first addend as the integer case for $m = 1$. For the second, in $K \cup K_2 f\varphi = \varphi r_1$ and

$$\partial(\varphi r_1) = \partial r_1 \varphi + r_1 \partial\varphi, \tag{10}$$

$$\bar{\partial}(\varphi r_1) = r_1 \bar{\partial}\varphi. \tag{11}$$

Let $h = r_1 \partial\varphi$. As the previous case, we consider the following situations.

(i) If $x, y \in K$, then

$$\|r_1 \partial\varphi\|_{\alpha, K} \leqslant \|\partial\varphi\|_{\alpha, \mathbb{C}} \|r_1\|_{\alpha, K} \leqslant M \|r_1\|_{m, K}.$$

(ii) If $x, y \in K_2$, $h \equiv 0$, therefore $|h(x) - h(y)|/|x - y|^\alpha = 0$.

(iii) Finally, if $x \in K$ and $y \in K_2$, we have

$$\frac{|h(x) - h(y)|}{|x - y|^\alpha} = \frac{h(x)|}{|x - y|^\alpha}.$$

Then we may consider that $x \in U_2^c \cap K$, since otherwise $h(x) = 0$, and besides that there exists a $w \in \bar{U}$ such that $|x - w| \leqslant |x - y|$. Hence, as above h vanishes at some point u of $K \cap E$,

$$\frac{|h(x) - h(y)|}{|x - y|^\alpha} \leqslant M\left(1 + \left(1 + \frac{d}{\vartheta}\right)^\alpha\right) \|r_1\|_{\alpha, K} \leqslant M \|r_1\|_{m, K},$$

where $\vartheta = \text{dist}(U_2^c, K_2)$. The last estimation follows from Proposition 3, and similar estimations follow for $h = \partial r_1 \varphi$ and $h = r_1 \bar{\partial}\varphi$.

4. C^m-Approximation by Meromorphic Functions

So far we have approximated functions defined on a compact set $K \subset \mathbb{C}$, and the approximating functions have been rational functions. Now we approximate functions defined on a closed set F in a domain G. The next theorem reduces the problem of approximating f on F by meromorphic functions in C^m norms to the problem of approximating functions on compact sets by rational functions in C^m norms. It is important to observe that if f belongs to $M_F(G)$, then $f_{|F}$ may be extended to a function in $\mathscr{C}^m(G)$. We need to recall the following definition.

Definition. Let D be a domain of $\mathbb{C}$. An exhaustion of D by precompact domain is a sequence $\{D_n\}_{n=1}^{\infty}$ of bounded subdomains of D satisfying

$$\bar{D}_n \subset D_{n+1}$$

$$\bigcup_{n=1}^{\infty} D_n = D$$

Theorem 5. *Let f be a complex-valued function defined on relatively closed subset F of a domain G of the complex plane. Then $f \in [M_F(G)]_{m,F}$ if, and only if, for every G_n*

$$f_{|F \cap \bar{G}_n} \in [R(F \cap \bar{G}_n]_{m, F \cap \bar{G}_n}, \tag{12}$$

where $\{G_n\}_{n=1}^{\infty}$ is some exhaustion of G by precompact domains.

Proof. It is easy to prove that (12) is necessary if f can be approximated on F by functions $g \in M_F(G)$ in C^m norms. In fact, each such g is analytic on $F \cap \bar{G}_n$ and f can be approximated on $F \cap \bar{G}_n$ in C^m-norms by rational functions by the C^m–Runge Theorem.

Now let us suppose that $f_{|F \cap \bar{G}_n} \in [R(F \cap \bar{G}_n]_{m, F \cap \bar{G}_n}$ where $\{G_n\}_{n=1}^{\infty}$ is an exhaustion of G by precompact domains. For each n we can apply the fusion lemma with

$$K_1 = \bar{G}_n,$$

$$K_2 = \mathbb{C}^* \backslash G_{n+1},$$

and

$$F_n = F \cap \bar{G}_{n+1}.$$

This yields constants A_n that we may suppose increasing and $A_n \geqslant 1$.

By hypothesis $f_{|F_n} \in [R(F_n)]_{m, F_n}$ for each n, and given $\varepsilon > 0$ we can find a rational function q_n without poles on F such that

$$\|f - q_n\|_{m, F_n} < \frac{\varepsilon}{2^{n+1} A_n} \qquad n = 1, 2, ..., . \tag{13}$$

Since $F_n \subset F_{n+1}$, $n = 1, 2, ...$, we have

$$\|q_{n+1} - q_n\|_{m, F_n} < \frac{\varepsilon}{2^n A_n}.$$

By Theorem 4 (the fusion lemma in C^m norms), for each n there exists a rational function r_n satisfying

$$\|r_n - q_n\|_{m, F_n \cup K_1} < \frac{\varepsilon}{2^n} \tag{14}$$

and

$$\|r_n - q_{n+1}\|_{m, F_n \cup K_2} < \frac{\varepsilon}{2^n}. \tag{15}$$

With these rational functions r_n and q_n we define

$$g = q_1 + \sum_{n=1}^{\infty} (r_n - q_n).$$

Note that (14) implies that for fixed n and $z \in G_n$, the function $r_k - q_k$ is analytic in G_n as soon as $k \geqslant n$, and it also guarantees the uniform convergence of $\sum_{k \geqslant n} (r_n - q_n)$ on G_n, so that g is analytic in G_n with the exception of finitely many poles. Hence $g \in M_F(G)$.

Finally it only remains to prove that

$$\|f - g\|_{m, F} < \varepsilon$$

or, in the other words,

$$\|f - g\|_{m, F_n} < \varepsilon \qquad n = 1, 2, ...$$

For $n = 1$,

$$\|f - g\|_{m, F_1} \leqslant \|q_1 - f\|_{m, F_1} + \sum_{n=1}^{\infty} \|r_n - q_n\|_{m, F_1} < \varepsilon,$$

and for $n > 1$

$$\|f-g\|_{m, F_n} \leqslant \sum_{k=1}^{n-1} \|r_k - q_{k+1}\|_{m, F_n} + \|q_n - f\|_{m, F_n} + \sum_{k=n}^{\infty} \|r_n - q_n\|_{m, F_n}$$

$$= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})$$

Since $F_n \subset (\mathbb{C}^* \backslash G_{k+1}) \cup F_k$, if $1 \leqslant k \leqslant n-1$, it follows from (15) that

$$(\mathrm{I}) = \sum_{k=1}^{n-1} \|r_k - q_{k+1}\|_{m, F_n} \leqslant \sum_{k=1}^{n-1} \frac{\varepsilon}{2^k}$$

and from (13)

$$(\mathrm{II}) = \|q_n - f\|_{m, F_n} \leqslant \frac{\varepsilon}{2^{n+1}}.$$

For $k \geqslant n$, where $F_n \subset F_k \cup \bar{G}_n$, and from (14) we obtain

$$(\mathrm{III}) \leqslant \sum_{k=n}^{\infty} \frac{\varepsilon}{2^k}$$

and the theorem is proved.

This theorem characterizes those functions that can be approximated in C^m norms by meromorphic functions on a closed set, and as a consequence we can extend Theorem 1 to closed subsets of the plane.

COROLLARY 6. *Let Ω be a domain of $\mathbb{C}$, and F a relatively closed subset of Ω. Then every function of $\mathscr{C}^m(\Omega)$ can be approximated in C^m norms by functions of $M_F(\Omega)$ if, and only if,*

$$\partial^i \bar{\partial}^{j+1} f(z) = 0 \qquad 0 \leqslant i + j \leqslant [m] - 1$$

for all $z \in F$.

Proof. Suppose that $\partial^i \bar{\partial}^{j+1} f(z) = 0$ $\forall z \in F$, $0 \leqslant i + j \leqslant [m] - 1$. Trivially, this holds for every $z \in F \cap \bar{G}_n$, where G_n is a bounded subdomain of Ω. Thus $f_{|F \cap \bar{G}_n}$ may be approximated by rational function in C^m norms on $F \cap \bar{G}_n$ ([13, 9]), and by Theorem 5, $f \in M_F(\Omega)$. The other implication is clear.

The C^m–Runge Theorem for arbitrary closed subset of $\mathbb{C}$ follows immediately from the above corollary, but also we can prove it by using the localization theorem and Runge's theorem for a compact subset.

COROLLARY 7. *Let F be a relatively closed subset of Ω and $f \in H(F)$, then f can be approximated in C^m-norms by functions in $M_F(\Omega)$.*

Proof. Choose an exhausting sequence of Ω by precompact domains $(G_n)_{n=1}^{\infty}$. If F is a closed subset of $\mathbb{C}$ and $f \in H(F)$ then $f \in H(F \cap \bar{G}_n)$ for every $n = 1, 2, \ldots$. Hence, by Theorem 2 (C^m-Runge Theorem) $f \in [R(F \cap \bar{G}_n)]_{m, F \cap \bar{G}_n}$, and Theorem 5 implies that $f \in [M_F(G)]_{m, F}$.

Observe that Corollary 6 is also a consequence of Corollary 7 and Theorem 2.1.1 in [3]

5. C^m-APPROXIMATION BY ANALYTIC FUNCTIONS

In the context of uniform approximation, Arakeljan [1] has characterized the sets of holomorphic approximation by imposing conditions on F and Ω. One procedure to prove Arakeljan's Theorem is to approximate by meromorphic functions and to move the poles of these functions to the boundary of Ω. This technique was employed in [4] in order to extend Arakeljan's Theorem to Lip α. Now, we obtain a similar theorem in C^m norms; in this case to assure the meromorphic approximation we also must impose other conditions on the function to be approximated.

We need first to establish some auxiliary results, whose proofs are similar to the ones of Lemmas 8 and 9 and Theorem 11 in [4], by considering C^m norms instead of Lip α norms.

LEMMA 8. *Suppose z_1 and z_2 lie in the same connected component of $\Omega \setminus F$ and $f = P(1/(z - z_1))$ where P is a polynomial. Then for every $\varepsilon > 0$ there exists a polynomial P_1 fulfilling*

$$g = P_1\left(\frac{1}{z - z_2}\right)$$

and

$$\|f - g\|_{m, F} < \varepsilon.$$

LEMMA 9. *Let Ω be a domain in $\mathbb{C}$, F be a relatively closed set in Ω, and $f \in M(\Omega)$. Suppose that f has a pole at $z_1 \in \Omega \setminus F$. If $\varepsilon > 0$ and $z_2 \in \Omega \setminus F$ with z_1 and z_2 being in the same connected component of $\Omega \setminus F$, then there exists a function $g \in M(\mathbb{C})$ such that:*

(a) *g has a pole at z_2.*

(b) *g is holomorphic at z_1.*

(c) *If z is a pole of g and $z \neq z_i$, for $i = 1, 2$, then z is also a pole of f.*

(d) $\|f - g\|_{m, F} < \varepsilon$.

In Section 2 we saw that the poles of the approximating rational functions can be fixed. In other words, it can be advantageous to recolocate the poles of the approximating rational functions without affecting the approximation itself. So it is important that in C^m-norms an analogous result holds for meromorphic functions, since it would allow us to move the poles of the approximating meromorphic functions to the boundary of Ω (or ∞ if $\Omega = \mathbb{C}$) and approximation by holomorphic functions would be established. A basic result is a shifting-of-poles theorem in C^m-norm, which is established in the next theorem.

THEOREM 10. *Let Ω be a domain in $\mathbb{C}$, and F be a relatively closed set in Ω, $\Omega^* \backslash F$ being locally connected. Then for every $f \in M_F(\Omega)$ and $\varepsilon > 0$ there exists a rational function R and a holomorphic function h in Ω such that*

$$\|f - R - h\|_{m, F} < \varepsilon.$$

If $\Omega^* \backslash F$ is connected, R can be taken to be $\equiv 0$.

If we denote $A^m(F) = \{f \in \mathscr{C}^m(\Omega) \colon \partial^i \bar{\partial}^{j+1} f \equiv 0, \text{ on } F, 0 \leqslant i + j \leqslant [m] - 1\}$, then we can extend Arakeljan's theorem to C^m norms.

THEOREM 11. *$A^m(F) = [H(\Omega)]_{m, F}$ if, and only if, $\Omega^* \backslash F$ is connected and locally connected.*

Proof. The sufficiency of the conditions can be deduced from Theorem 5 and Theorem 10.

If $A^m(F)$ is approximated in C^m-norm by functions in $H(\Omega)$ then $\Omega^* \backslash F$ is connected. Otherwise, we can define a $f \in C^m(\Omega)$ such that $f_{|U} = 1/(z - z_0)$, with z_0 belonging to a bounded component of $\Omega^* \backslash F$ and U being a neighbourhood of F. Thus $f \in A^m(F)$. And f can be approximated in the C^m-norm by functions in $H(\Omega)$. In particular, we can approximate in uniform norm the function $1/(z - z_0)$ on F by functions in $H(\Omega)$ and we obtain a contradiction with the maximum principle [5, p. 142].

If $\Omega^* \backslash F$ is not locally conected, then there exist a neighbourhood $U = \Omega \backslash K$ (K compact in Ω) of $\{*\}$ and a sequence $z_n \in \Omega \backslash F$, $z_n \to \{*\}$, that cannot be connected in $U \backslash F$ with $\{*\}$. With the help of the Mittag-Leffler theorem, we construct a function f meromorphic in Ω having simple poles at the points z_n with residues nd_n. Then by Whitney's theorem we can consider that $f \in A^m(F)$, and f can be approximated in C^m-norm by $H(\Omega)$. Thus we also obtain a contradiction [5, p. 142].

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